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ACTIVATION FUNCTIONS OF GENERALIZED (L)-TYPE IN NEURAL NETWORKS THEORY

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Abstract. In this work we enlarge the class of admissible activation functions, in a Hpfield neural network system, leading to exponential stability. Most of the existing treatments so far consider only Lipschitz continuous activation functions. Here, we discuss the case of functions of "generalized" (L)-type which are not necessarily Lipschitz continuous and systems with variable coefficients. The case of constant coefficients is given as a particular case.

Keywords: Neural network, activation function, (L)-type kernel, exponential stabilization.
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1 Introduction

Prior to any decision, oil and gas engineers must go through a long process of checking whether it is worthy or not exploring a given field. They need to have an idea about the amount of hydrocarbon present in the reservoir and how it can be extracted. To this end, it is necessary to have an insight on different reservoir parameters such as the permeability and the porosity. In the last three decades, scientists and engineers have been focusing on the use of Artificial Intelligence to investigate reservoir characteristics and they have been very successful in that. Artificial Neural Networks (combined with other techniques) have been use to predict some properties from well log data (Bhatt & Helle, 2002; Majdi et al., 2010; Verma et al., 2014), to characterize fractured reservoirs (Adibifard et al., 2014; El Ouahed, 2005), to determine lithofacies and lithology (Bohling & Dubois, 2002; Negi, 2006) and in history matching (Maschio et al., 2010; Silva et al., 2007).

In addition to providing highly accurate results, in comparison with the conventional single-processor machines, ANNs are fault tolerant, can identify new objects and can predict future events.

A simple system which occurs in the theory of Neural Networks is the following

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} f_{ij}\left(t, x_j(t)\right) + I_i(t),\ t \geq 0,\ i = 1, ..., m \tag{1}$$

with given initial data $x_i(0) = x_{i0}$, $i = 1, ..., m$. It is known under the name: Hopfield neural network. The functions $I_i(t)$, $i, j = 1, ..., m$ are real-valued continuous functions and $a_i(t)$, $i = 1, ..., m$ are nonnegative continuous functions. The functions f_{ij} are the activation functions.

Artificial Neural Network (ANN) is an information processing device that has a broad spectrum of application. Some of the applications are: target marketing, consumer behavior analysis, credit scoring, strategic planning and performance, data mining, price forecast, physical system modelling, soil behavior, soil swelling, speech processing, traveling salesman, pattern recognition (Bouzerdoum & Pattison, 1993; Chua & Roska, 1990; Crespi, 1999; Hopfield, 1982; Hopfield & Tank, 1986; Inoue, 1996; Kennedy & Chua, 1998; Kosko, 1991; Mohamad, 2007; Qiao et al., 2001; Sudharsanan & Sundareshan, 1991; Van den Driessche & Zou, 1998; Wu & Xue, 1996).

In contrast to single-processor computers (based on the von Neumann architecture), which perform their tasks in sequence, ANNs are composed of many processors which are highly connected and work in parallel. A processor receives signals from the input layer, processes them using a transfer function and then sends the processed information to the subsequent layer.

Apart from some papers, see for instance (Forti et al., 2005; Forti & Tesi, 1995; Huang et al., 2009; Tatar, 2012, 2014a,b,c,d, 2015a,b, 2017; Wu, 2009; Wu et al., 2011; Wu & Xue, 2008) all the other existing works in the literature assume that the activation functions are at least Lipschitz continuous (Bouzerdoum & Pattison, 1993; Chua & Roska, 1990; Crespi, 1999; Hopfield, 1982; Hopfield & Tank, 1986; Inoue, 1996; Kennedy & Chua, 1998; Kosko, 1991; Mohamad, 2007; Qiao et al., 2001; Sudharsanan & Sundareshan, 1991; Van den Driessche & Zou, 1998; Wu & Xue, 1996). In applications, activation functions which do not satisfy this condition are not rare (see (Kosko, 1991)). Therefore, it is of great importance to have some insight about these cases too. It is also worth noting that most of the works on variable coefficients treat rather the stability of periodic solutions (Huang et al., 2009; Song, 2008).

Here (for simplicity) we consider activation functions f_{ij} satisfying

$$\left|f_{ij}\left(t,x\right)-f_{ij}\left(t,y\right)\right|\leq b_{ij}(t)L_{j}\left(t,\left|x-y\right|\right),\ i,j=1,...,m \tag{2}$$

(instead of two subscripts in L_{ij}) and L_j are nonlinearities of "generalized" (L)-type i.e such that

$$0\leq L_j(t,x)-L_j(t,y)\leq M_j(t,y)g_j(x-y),\ t\in I,\ x\geq y\geq 0,\ j=1,...,m \tag{3}$$

where g_j are nondecreasing continuous functions on $\mathbb{R}_+=[0,+\infty)$ with $g_j(u)>0$ for $u>0$ and $M_j : I\times\mathbb{R}_+\to\mathbb{R}_+$, $j=1,...,m$ are nondecreasing continuous functions. The case $g_j(u)=u$, $j=1,...,m$ corresponds to the (L)-type nonlinearities introduced in Dragomir (1987). In fact, we will need to generalize a lemma in Dragomir (1987) to prove our result.

In the next section we prepare some material and give two lemmas which will be used later. Section 3 contains the statement and proof of our result. In Section 4 we treat the constant coefficients case as a special case and provide some numerical simulations illustrating our result.

2 Preliminaries

In case of time-independent coefficients and functions, the "equilibrium" is denoted by $(x_1^*,..., x_m^*)^T$. It is solution of

$$0=-a_ix_i^*+\sum_{j=1}^{m}f_{ij}\left(x_j^*\right)+I_i,\ i=1,...,m,\ t\geq 0.$$

In case of time-dependent coefficients and functions, we may assume that $(x_1^*,..., x_m^*)^T$ is any final (desired) state we want to reach. We introduce the controls $u_i(t)$ by

$$0=-a_i(t)x_i^*+\sum_{j=1}^{m}f_{ij}\left(t,x_j^*\right)+I_i(t)+u_i(t),\ i=1,...,m,\ t\geq 0. \tag{4}$$

Here, we assume the existence of a unique equilibrium in the time-independent case. Otherwise, we consider the difference of any two solutions $x_i(t)$ and $x_i^*(t)$ and prove the convergence to zero (in an exponential manner).

Let $I \subset \mathbf{R}$, and let $g_1, g_2 : I \to \mathbf{R}\backslash\{\mathbf{0}\}$. We write $g_1 \propto g_2$ if g_2/g_1 is nondecreasing in I.

The first lemma is the Bihari-Gronwall inequality with several integrals (Pinto, 1990).

Lemma 1. *Let $k_j(t)$, $j = 1, ..., n$ be nonnegative continuous functions in $J := [\alpha, \beta]$, $g_j(u)$, $j = 1, ..., n$ are nondecreasing continuous functions in $\mathbb{R}_+$, with $g_j(u) > 0$ for $u > 0$, $a > 0$ and $u(t)$ is a nonnegative continuous functions in J. If $g_1 \propto g_2 \propto ... \propto g_n$ in $(0, \infty)$, then the inequality*

$$u(t) \leq a + \sum\nolimits_{j=1}^{n} \int_{\alpha}^{t} k_j(s) g_j(u(s)) ds, \ t \in J,$$

implies that

$$u(t) \leq G_n^{-1} \left[G_n(c_{n-1}) + \int_{\alpha}^{t} k_n(s) ds \right], \ \alpha \leq t \leq \beta_0$$

where $c_0 := a$,

$$c_j := G_j^{-1} \left[G_j(c_{j-1}) + \int_{\alpha}^{\beta_0} k_j(s) ds \right], \ j = 1, ..., n-1,$$

$$G_j(u) := \int_{u_j}^{u} \frac{dx}{g_j(x)}, \ u > 0 \ (u_j > 0, \ j = 1, ..., n),$$

G_j^{-1} is the inverse of G_j and β_0 is the largest number such that

$$\int_{\alpha}^{\beta_0} k_j(s) ds \leq \int_{c_{j-1}}^{u} \frac{dx}{g_j(x)}, \ j = 1, ..., n.$$

This lemma is generalized, for instance, to $a = a(t)$ in Pinto (1990). It has been shown, in particular, that the $c_j's$ and the bound on $u(t)$ do not depend on the choice of (u_j in) $G_j(u)$.

Next, we introduce and prove a lemma which is in fact similar to Dragomir's Lemma (Dragomir, 1987) but with general functions g_j instead of the identity.

Lemma 2. *Let $u : I = [\alpha, \beta] \to \mathbb{R}_+$ be a continuous function, $a > 0$ and $b(t)$ a strictly positive continuously differentiable function. Assume that $L_j : I \times \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, ..., m$ are continuous functions satisfying*

$$0 \leq L_j(t, x) - L_j(t, y) \leq M_j(t, y) g_j(x - y), \ t \in I, \ x \geq y \geq 0, \ j = 1, ..., m$$

where $M_j : I \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, ..., m$ are nondecreasing continuous functions satisfying $g_j(u) > 0$ for $u > 0$ and $g_1 \propto g_2 \propto ... \propto g_m$. Moreover, let $g_0(u) = u$. If

$$u(t) \leq a + b(t) \sum\nolimits_{j=1}^{m} \int_{\alpha}^{t} L_j(s, u(s)) ds, \ t \in I,$$

then

$$u(t) \leq a + b(t) \sum\nolimits_{j=1}^{m} \int_{\alpha}^{t} L_j(s, a + c_m(s)) ds,$$

on some sub-interval $[\alpha, \beta_0]$, with

$$c_j(t) := G_j^{-1} \left[G_j(c_{j-1}(t)) + \int_{\alpha}^{t} k_j(s) ds \right], \ j = 0, 1, ..., m,$$

where

$$c_{-1}(t) := \sum\nolimits_{j=1}^{m} \int_{\alpha}^{t} b(s) L_j(s, a) ds, \ k_0(s) := \frac{b'(s)}{b(s)},$$

$$k_j(s) := b(s)M_j(s,a),\ j = 1, ..., m,$$

and

$$G_j(u) := \int_{u_j}^{u} \frac{dx}{g_j(x)},\ u > 0\ (u_j > 0,\ j = 0, 1, ..., m).$$

Proof. Let $z(t) = b(t)\sum_{j=1}^{m}\int_{\alpha}^{t} L_j(s,u(s))ds$, $t \in I$. Then $z(\alpha) = 0$,

$$u(t) \le a + z(t),\ t \in I$$

and, adding and subtracting the expression $b(t)\sum_{j=1}^{m} L_j(t,a)$

$$\begin{gathered} z'(t) = b'(t)\sum_{j=1}^{m}\int_{\alpha}^{t} L_j(s,u(s))ds + b(t)\sum_{j=1}^{m} L_j(t,u(t)) \\ \le \frac{b'(t)}{b(t)}b(t)\sum_{j=1}^{m}\int_{\alpha}^{t} L_j(s,u(s))ds \\ +b(t)\sum_{j=1}^{m} L_j(t,a+z(t)) - b(t)\sum_{j=1}^{m} L_j(t,a) + b(t)\sum_{j=1}^{m} L_j(t,a),\ t \in I. \end{gathered}$$

By our assumption on L_j, we may write

$$z'(t) \le \frac{b'(t)}{b(t)}z(t) + b(t)\sum_{j=1}^{m} M_j(t,a)g_j(z(t)) + b(t)\sum_{j=1}^{m} L_j(t,a),\ t \in I. \tag{5}$$

Therefore, by integrating both sides of (5), we see that

$$z(t) \le \sum_{j=1}^{m}\int_{\alpha}^{t} b(s)L_j(s,a)ds + \int_{\alpha}^{t}\left[\frac{b'(s)}{b(s)}z(s) + b(s)\sum_{j=1}^{m} M_j(s,a)g_j(z(s))\right]ds$$

or in short

$$z(t) \le A(t) + \sum_{j=0}^{m}\int_{\alpha}^{t} k_j(s)g_j(z(s))ds$$

with

$$A(t) := \sum_{j=1}^{m}\int_{\alpha}^{t} b(s)L_j(s,a)ds$$

$g_0(v) := v$ and $k_0(\sigma) = \frac{b'(\sigma)}{b(\sigma)}$, $k_j(\sigma) = b(\sigma)M_j(\sigma,a)$, $j = 1, ..., m$. An application of Lemma 1 yields

$$u(t) \le a + z(t) \le a + c_m(t),\ \alpha \le t < \beta_0$$

where $c_m(t)$ is as in the statement of the theorem. □

The strict positivity of $b(t)$ is not restrictive as we can always add a positive constant to $b(t)$. Also, we can consider $a = a(t)$ with $a(t)$ nondecreasingness or use $\tilde{a}(t) = \max a(t)$ instead of $a(t)$.

2.1 The exponential decay

In this section we state and prove our result. We denote by $y_i(t) = x_i(t) - x_i^*$, $y(t) = \sum\limits_{i=1}^{m} |y_i(t)|$, $a(t) := \min_{1\le i\le m}\{a_i(t)\}$ and $b_i(t) := \sum\limits_{j=1}^{m} L_i\,|b_{ji}(t)|$, $i = 1, ..., m$ where L_i are defined in (2).

Theorem 1. *Assume that f_{ij} satisfy (2)-(3), $i, j = 1, ..., m$ and M_j are continuous nondecreasing functions. Then, there exists a $\beta_0 > 0$ such that*

$$\begin{aligned} &y(t) \le \exp\left(-\int_0^t a(s)ds\right) \\ &\times\left[y(0) + \sum_{j=1}^{m}\int_0^t\left(\sum_{i,j=1}^{m}\tilde{b}_{ij}(s)\right)L_j(s,y(0)+c_m(s))ds\right],\ 0 \le t < \beta_0 \end{aligned}$$

where

$$\tilde{b}_{ij}(t) := b_{ij}(t) \exp \int_0^t a(s)ds,\ c_0(t) := \sum\nolimits_{j=1}^m \int_\alpha^t \left(\sum_{i=1}^m \tilde{b}_{ij}(s) \right) L_j(s, y(0))ds,$$

$$k_j(s) := \left(\sum_{i=1}^m \tilde{b}_{ij}(s) \right) M_j(s, y(0)),\ j = 1, ..., m,$$

$$c_j(t) := G_j^{-1} \left[G_j \left(c_{j-1}(t) \right) + \int_0^t k_j(s)ds \right],\ j = 1, ..., m,$$

and

$$G_j \left(u \right) := \int_{u_j}^u \frac{dx}{g_j(x)},\ u > 0\ (u_j > 0,\ j = 1, ..., m).$$

Proof. It is not difficult to see, from the system and our assumptions, that

$$D^+ \left| y_i(t) \right| \leq -a_i(t) \left| y_i(t) \right| + \sum_{j=1}^m b_{ij}(t) L_j \left(t, \left| y_j(t) \right| \right),\ t > 0,\ i = 1, ..., m$$

or

$$\begin{aligned} D^+ y(t) &\leq -\min_{1 \leq i \leq m} \left\{ a_i(t) \right\} y(t) + \sum_{i,j=1}^m b_{ij}(t) L_j \left(t, y(t) \right) \\ &\leq -a(t) y(t) + \sum_{i,j=1}^m b_{ij}(t) L_j \left(t, y(t) \right),\ t > 0 \end{aligned} \tag{6}$$

where D^+ denotes the right Dini derivative. Therefore, the relation (6) implies

$$\begin{aligned} D^+ \left[y(t) \exp \textstyle\int_0^t a(s)ds \right] &\leq \sum_{i,j=1}^m b_{ij}(t) L_j \left(t, y(t) \right) \exp \left(\textstyle\int_0^t a(s)ds \right) \\ &\leq \sum_{i,j=1}^m \tilde{b}_{ij}(t) L_j \left(t, y(t) \right),\ t > 0 \end{aligned} \tag{7}$$

where $\tilde{b}_{ij}(t) := b_{ij}(t) \exp \left(\int_0^t a(s)ds \right)$, and by comparison and integration we entail from (7) that

$$y(t) \exp \left(\int_0^t a(s)ds \right) \leq y(0) + \sum_{i,j=1}^m \int_0^t \tilde{b}_{ij}(s) L_j \left(s, y(s) \right) ds$$

or

$$\tilde{y}(t) \leq y(0) + \sum_{i,j=1}^m \int_0^t \tilde{b}_{ij}(s) L_j \left(s, y(s) \right) ds,\ t > 0$$

where $\tilde{y}(t) := y(t) \exp \left(\int_0^t a(s)ds \right)$. Note that $L_j \left(t, y(t) \right) \leq L_j \left(t, \tilde{y}(t) \right)$ because $0 \leq L_j \left(t, x \right) - L_j \left(t, y \right)$ when $x \geq y$ by definition and assumptions. Hence

$$\tilde{y}(t) \leq y(0) + \sum_{j=1}^m \int_0^t \left(\sum_{i,j=1}^m \tilde{b}_{ij}(s) \right) L_j \left(s, \tilde{y}(s) \right) ds,\ t > 0 \tag{8}$$

and we can apply Lemma 2 to (8) to obtain

$$\tilde{y}(t) \leq y(0) + \sum_{j=1}^m \int_0^t \left(\sum_{i=1}^m \tilde{b}_{ij}(s) \right) L_j(s, y(0) + c_m(s))ds,\ 0 \leq t < \beta_0$$

for some $\beta_0 > 0$ where

$$c_0(t) := \sum\nolimits_{j=1}^{m} \int_{\alpha}^{t} \left(\sum_{i=1}^{m} \tilde{b}_{ij}(s) \right) L_j(s, y(0))ds,$$

$$k_j(s) := \left(\sum_{i=1}^{m} \tilde{b}_{ij}(s) \right) M_j(s, y(0)),\ j = 1, ..., m,$$

$$c_j(t) := G_j^{-1} \left[G_j \left(c_{j-1}(t) \right) + \int_0^t k_j(s)ds \right],\ j = 1, ..., m,$$

and

$$G_j\left(u\right) := \int_{u_j}^{u} \frac{dx}{g_j(x)},\ u > 0\ (u_j > 0,\ j = 1, ..., m).$$

The proof is complete. □

Corollary 1. *In case β_0 is infinite we have global existence. This occurs when*

$$\int_0^{\infty} k_j(s)ds \leq \int_{\omega_{j-1}}^{\infty} \frac{d\sigma}{g_j(\sigma)},\ j = 1, ..., m.$$

Remark. In the situation of the corollary, the asymptotic stability has the rate

$$\exp\left(-\int_0^t a(s)ds \right) \sum_{j=1}^{m} \int_0^t \left(\sum_{i=1}^{m} \tilde{b}_{ij}(s) \right) L_j(s, y(0) + c_m(s))ds.$$

That is, solutions approach the equilibrium at an exponential rate if $\int_0^t a(s)ds \to \infty$ *as* $t \to \infty$ and

$$\sum_{j=1}^{m} \int_0^t \left(\sum_{i,j=1}^{m} \tilde{b}_{ij}(s) \right) L_j(s, c_m(s))ds$$

grows up at an exponential rate smaller than $\exp\left(\int_0^t a(s)ds \right)$.

In case of control, the controls $u_i(t)$ (see (4)) are able to drive solutions of (1) to the prescribed state $(x_1^*, ..., x_m^*)^T$ exponentially.

2.2 Constant coefficients case

Special case: We consider Problem (1) with constant coefficients. That is, $a_i(t) \equiv a_i > 0$, $c_i(t) \equiv c_i$ and instead of $f_{ij}\left(t, x_j(t)\right)$, we consider $b_{ij} f_j\left(x_j(t)\right),\ i, j = 1, ..., m$. We obtain the system

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j\left(x_j(t)\right) + c_i,\ i = 1, ..., m.$$

The equilibrium is defined through the system

$$0 = -a_i x_i^* + \sum_{j=1}^{m} b_{ij} f_j\left(x_j^*\right) + c_i,\ i = 1, ..., m.$$

It is assumed to exist and is unique. The functions f_j are assumed to satisfy

$$\left| f_j\left(x\right) - f_j\left(y\right) \right| \leq L_j\left(|x - y| \right),\ j = 1, ..., m$$

and L_j are nonlinearities of "generalized" (L)-type i.e such that

$$0 \leq L_j(x) - L_j(y) \leq M_j(y) g_j(x - y),\ x \geq y \geq 0,\ j = 1, ..., m$$

for some continuous and nondecreasing functions M_j and g_j, $j = 1, ..., m$. The same argument as in the proof of Theorem 1 shows that

$$y(t) \leq e^{-at} \left[y(0) + \sum_{j=1}^{m} \int_0^t \left(\sum_{i=1}^{m} \tilde{b}_{ij}(s) \right) L_j(y(0) + c_m(s)) ds \right], \ t > 0$$

where $\tilde{b}_{ij}(s)$, L_j and c_m are as in Theorem 1 with time independent coefficients. Observe here that $\tilde{b}_{ij}(s)$ and $c_m(s)$ are of the order e^{-as}. The best we can get is a bound.

In the next section we consider this case together with the case of variable b_{ij} of the form $b_{ij}(t) = e^{-b_{ij}t}$ with $b_{ij} > 0$ to get a decay of this order.

3 Numerical study

In this section, we present numerical examples to illustrate our exponential stability result for non-Lipschitz activation functions. We consider the following two dimensional, constant coefficients system (the exponentials are parts of the functions f_{ij}):

$$\begin{cases} x_1'(t) = -a_1 x_1(t) + c_{11} e^{-\beta_{11} t} x_1^{\gamma_{11}}(t) + c_{12} e^{-\beta_{12} t} x_2^{\gamma_{12}}(t), \ t \geq 0 \\ x_2'(t) = -a_2 x_2(t) + c_{21} e^{-\beta_{21} t} x_1^{\gamma_{21}}(t) + c_{22} e^{-\beta_{22} t} x_2^{\gamma_{22}}(t), \ t \geq 0 \end{cases}$$

with $\beta_{ij} \geq 0$, $i, j = 1, 2$ and subject to initial data $[x_1(0), x_2(0)]^T$. The exponents γ_{ij} are chosen within the interval $(0, 1)$ to make the activation functions non-Lipschitz,

$$f_{ij}(t, x) := c_{ij} e^{-\beta_{ij} t} x_j^{\gamma_{ij}}(t), \ t \geq 0.$$

We numerically treat the above system of nonlinear ordinary differential equations using the fourth order Runge-Kutta method (RK4). First, let us show that the above system satisfies the required stability conditions (2)-(3). Notice that

$$\begin{aligned} |f_{ij}(t, x_1) - f_{ij}(t, x_2)| &= c_{ij} e^{-\beta_{ij} t} |x_1^{\gamma_{ij}}(t) - x_2^{\gamma_{ij}}(t)| \\ &\leq c_{ij} e^{-\beta_{ij} t} |x_1(t) - x_2(t)|^{\gamma_{ij}}, \ i, j = 1, 2, \ t \geq 0. \end{aligned}$$

Therefore,

$$b_{ij}(t) := c_{ij} e^{-\beta_{ij} t} \ \text{ and } \ L_{ij}(t, x) := |x|^{\gamma_{ij}}, \quad t \geq 0.$$

Also, for $x_1 \geq x_2$,

$$\begin{aligned} 0 &\leq L_{ij}(t, x_1) - L_{ij}(t, x_2) = |x_1|^{\gamma_{ij}} - |x_2|^{\gamma_{ij}} \\ &\leq (x_1 - x_2)^{\gamma_{ij}} =: g_{ij}(x_1 - x_2), \quad t \geq 0. \end{aligned}$$

Hence, conditions (2)-(3) are satisfied. Since $a = \min\{a_1, a_2\}$, we obtain

$$\tilde{b}_{ij}(t) = b_{ij} e^{at}, \ t \geq 0$$

and taking $M_1 = M_2 = 1$, it follows that

$$k_j(t) = \tilde{b}_{1j}(t) + \tilde{b}_{2j}(t) = (b_{1j} + b_{2j}) e^{at}, \quad t \geq 0.$$

Clearly, from the definition of g_{ij}, the system has a global solution since

$$\int_0^\infty \frac{d\sigma}{g_{ij}(\sigma)} = \infty, \ i, j = 1, 2.$$

Now, we present two applications of the above system which illustrate the convergence to the equilibrium.

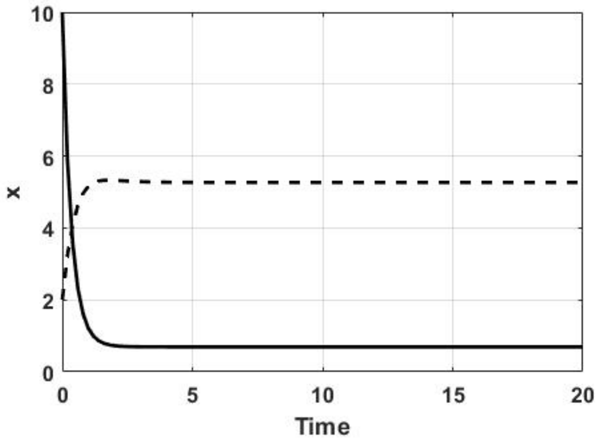

Figure 1: The dash- and solid-lines are the graphs of functions $x_1(t)$ and $x_2(t)$, respectively

Application 1: In this example, we take the following parameters: $a_1 = 2$, $a_2 = 3$, $\beta_{ij} = 0$, $c_{ij} = 1$, $i, j = 1, 2$ and

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 \\ 0.3 & 0.4 \end{bmatrix}, \quad \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 0.5 \end{bmatrix}$$

Fig. 1 shows that the numerical solution x $= [x_1(t), x_2(t)]^T$ converges to the equilibrium state $x^* = [5.2670, 0.6926]^T$.

Application 2: In this example, we consider the same inputs but with nonzero β_{ij},

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.25 \\ 0.3 & 0.4 \end{bmatrix}$$

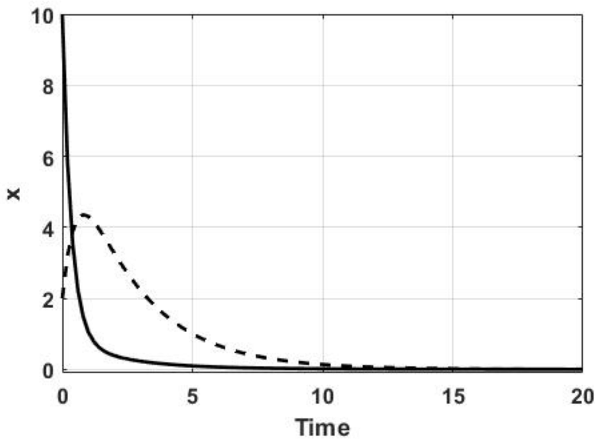

Figure 2: The dash- and solid-lines are the graphs of functions $x_1(t)$ and $x_2(t)$, respectively

Fig. 2 shows the convergence of the solution to zero.

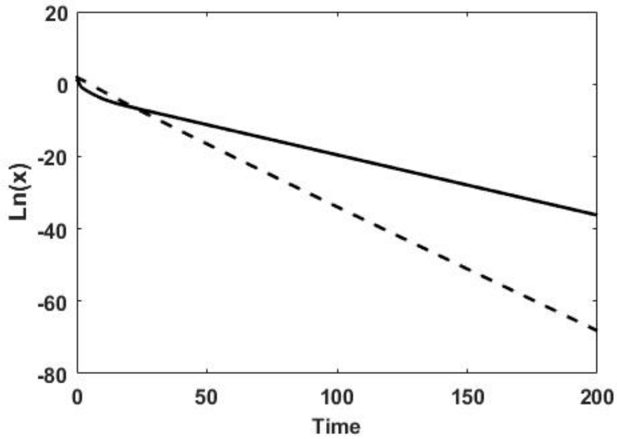

Figure 3: The dash- and solid-lines are the graphs of functions $ln\,(x_1(t))$ and $ln\,(x_2(t))$, respectively

Fig. 3 proves that $\ln(x_i)$ decays linearly, and hence the solution is exponentially stable.

In conclusion, the above numerical applications agree with our exponential stability results.

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